Strategic Games

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1 Nash Equilibrium

NE predicts the steady state behaviour of a set of populations – one for each player – playing the game repeatedly against a randomly selected anonymous opponent each time.

An NE embodies a 'stable social norm': if everyone else adheres to it, none will wish to deviate from it.[1]

1.1 Definition

α^* is mixed-strategy NE iff:

$$U_i(\alpha^*) \geq U_i(\alpha_i, \alpha^*_{-i}) \text{ for } \forall \alpha_i, \forall i \in N$$

a^* is a **strict** NE iff:

$$U_i(a^*) > U_i(a_i, a^*_{-i}) \text{ for } \forall a_i \neq a^*_i, \forall i \in N$$

1.2 Finding pure and mixed NE in games with discrete discrete strategy spaces

α^* is mixed-strategy NE iff for $\forall i \in N$:

$$\begin{aligned} &\nexists a_i \in A_i \text{ such that } U_i(a_i, \alpha^*_{-i}) > U_i(\alpha^*) \\ &\text{for } \forall a_i \text{ such that } U_i(a_i, \alpha^*_{-i}) < U_i(\alpha^*), \;\; \alpha^*_i(a_i) = 0 \end{aligned}$$

Procedure: Begin by reducing the game as much as possible using IESDS. Then: for each possible combination of actions (including single actions) of the row player,

1. *assume* that the row player strictly mixes over this combination;

[1] It may be the case, however, that some are able to deviate from it *without penalty*.

2. first consider if any actions of the column player can be eliminated straight away as not-best replies;
3. consider the restrictions that the row player *strictly mixing, as specified*, imposes on the column player's strategy weightings;[2] (– i.e. the restrictions needed to make *Row's* strategy a BR)
4. consider the restrictions that *these* values impose on the row player's strategy weightings (– i.e. the restrictions needed to make *Column's* strategy a BR);
5. you will either (i) isolate the equilibria in which the row player plays as describe, or (ii) arrive at a contradiction.
6. Finally: check to see if it is possible for the column player but not the row player to be mixing.

1.3 Finding pure NE in games with continuous strategy spaces

Procedure:

1. calculate each player's BR function;
2. if need be (e.g. if the BR functions are defined piecewise), divide one or more players' strategy space into cases;
3. for each case: assume that an NE exist in which the relevant player(s) strategies are within the subset selected by this case;
4. use the BR function to determine whether such an NE really is possible;
5. you will either (i) isolate the set of possible NE in this case, or (ii) arrive at a contradiction.

1.4 Existence of (Undominated) NE

If each player has finitely many actions,
then: there exists a mixed-strategy NE in which no player's strategy is weakly dominated.

[2]Use the theorem presented above.

1.5 Existence of Symmetric NE in Symmetric Games

If each player has finitely many actions and the game is symmetric,
then: there exists a symmetric mixed-strategy NE.

1.6 Choosing between NE

Some NE may be 'focal': because they are Pareto optimal; for aesthetic reasons; because of the properties of some unrelated game over the same variables.

Playing a weakly dominated strategy is unnecessarily 'risky': an NE with weakly dominated strategies is less plausible than one without one.

2 Rationality[3]

α_i is rational iff:

$$\exists \mu_i \text{ such that } \alpha_i \in BR_i(\mu_i)$$

3 Dominance

3.1 Strict Dominance

α_i is **strictly dominant** iff α_i strictly dominates a_i for $\forall a_i \in A_i \backslash \{\alpha_i\}$

a_i is **strictly dominated** iff $\exists \alpha_i$ such that α_i strictly dominates a_i.

α_i **strictly dominates** a_i iff:

$$U_i(\alpha_i, a_{-i}) > U_i(a_i, a_{-i}) \text{ for } \forall a_{-i} \in A_{-i}$$

3.2 Weak Dominance

a_i is **weakly dominated** iff $\exists \alpha_i$ such that α_i weakly dominates a_i.

α_i **weakly dominates** a_i iff:

$$\begin{aligned} & U_i(\alpha_i, a_{-i}) \geq U_i(a_i, a_{-i}) \text{ for } \forall a_{-i} \in A_{-i} \\ \text{and } \exists a_{-i} \in A_{-i} \text{ such that } & U_i(\alpha_i, a_{-i}) > U_i(a_i, a_{-i}) \end{aligned}$$

[3]Not to be confused with *rationalizability* – somewhat counter-intuitively, *rationality* is a *weaker* requirement than *rationalizability*.

4 Best Responses

4.1 to Actions

$$BR_i(a_{-i}) = \arg\max_{\alpha_i} U_i(\alpha_i, a_{-i})$$

The standard way to highlight BRs is to underline their payoff in the payoff matrix. When using this method in the exam, *draw out* the matrix, and *add the comment:* 'best responses are underlined'.

4.2 to Beliefs

$$BR_i(\mu_i) = \arg\max_{\alpha_i} \sum_{a_{-i} \in A_{-i}} \mu_i(a_{-i}) \cdot U_i(\alpha_i, a_{-i})$$

5 Never-Best Responses

α_i is a never-best response iff:

$$\alpha_i \notin BR_i(\mu_i) \text{ for } \forall \mu_i$$

In two person game, being a never-best response is equivalent to being strictly dominated.

This is true more generally iff our definition of beliefs allows for correlated conjectures.

6 Rationalizability

Rationalizability predicts the behaviour of players who have no experience playing the game, but who *attempt to deduce their opponents' rational actions from their opponents' preferences and from analyses of their opponents' reasoning about their own rational actions.*

Rationalizability is a weaker notion than NE ... but perhaps it is still too *strong*? Think of the numerical beauty contest .... Rationalizability requires it to be **common knowledge** that all players are rational.

(Rationalizability can also be defined in terms of iterated deletion of never-best replies.)

6.1 Definition

α_i^* is rationalizable iff: $\exists Z_1, \ldots, Z_n$ such that:

$$\begin{aligned}
&\text{for } \forall j,\ \ Z_j \subset A_j \\
&\text{for } \forall j,\ \ \text{for } \forall a_j \in Z_j,\ \ \exists \mu_j \ \text{ over } \ Z_{-j} \ \text{ such that: } \ a_j \in BR_j(\mu_j) \\
&\alpha_i^* \in Z_i
\end{aligned}$$

6.2 IESDS[4] and Rationalizability

If each player has finitely many actions,
then: a unique set of strategies survives IESDS – the set of rationalizable strategies.

7 IESDS

When performing IESDS in an exam, *draw out* the payoff matrix, and add *ordinal annotations ('1st', '2nd', '3rd', ...)* to your deletions.

[4] Not to be confused with *dominance solvability*.

8 Risk Dominance

In a coordination game

	H	G
H	A, a	C, b
G	B, c	D, d

(G, G) risk-dominates (H, H) iff:

$$(C - D)(c - d) \geq (B - A)(b - a)$$

8.1 Payoff Dominance

(H, H) payoff-dominates (G, G) iff it is Pareto superior.