

Extensive-Form Games: PBE, SE, and Forward Induction

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I Further Definitions

I.1 Completely Mixed Strategies

A completely mixed strategy for a player is a mixed strategy which assigns a strictly positive probability to each of the player's pure strategies.

I.2 Beliefs

Beliefs are defined as probability distributions over nodes in an information set.

Notation: we attach beliefs to extensive form diagrams by adding $\mu = c$ next to each node, with c varying from node to node.

I.3 Assessments

An assessment is an ordered pair (σ, μ) , where σ is a (potential mixed) strategy profile, and μ is a set of beliefs for all players at all information sets.

I.4 Sequential Rationality

An assessment (σ, μ) satisfies sequential rationality iff under σ each player i 's choice at each of her information sets is optimal given (1) her beliefs μ_i at the information set, and (2) all of the other players' strategies under σ .

I.5 Consistency

An assessment (σ, μ) is consistent iff there exists a sequence $\langle (\sigma^m, \mu^m) \rangle_{m=1}^{\infty}$ such that:

- (1) (σ^m, μ^m) converges to (σ, μ) as $m \rightarrow \infty$
- (2) σ^m is a completely mixed strategy profile for $\forall m$

(3) μ^m is computed from σ^m via Bayes rule for $\forall m$

Consistency is a stronger requirement than on-path-Bayesian-updating: but not by much! We're still allowed to construct the *pattern of mistakes* $\sigma^m(h)$ from $\sigma(h)$ (for any h) *however we like*.

We could refine the concept of consistency further by specifying a theory of mistakes, e.g.: one mistake cannot be more likely than another.

2 Perfect Bayesian Equilibrium

An assessment (σ, μ) is a PBE iff:

- [**sequential rationality**] (1) each σ_i is optimal given σ_{-i} and μ_i
(2) beliefs in μ on the equilibrium path are computable via Bayes rule
(... off-path beliefs can be chosen freely)

2.1 Existence of PBE

If (1) there are finitely many terminal histories,
and (2) every terminal history is finite, ('finite time horizon')
then: there exists at least one PBE.

2.2 Finding PBE

When finding PBE, it's usually easiest to find SPE first, and then consider which of them can be justified as PBE.

2.3 Critique

PBE places no restrictions on off-path beliefs.

This will be unreasonable if (*as is necessary to sustain SPE rather than a less rigorous solution concept* – Osborne p.163ff.) the steady state will always be perturbed somehow, so that nonequilibrium actions are taken.¹

Another reason why the failure to restrict off-path beliefs is unreasonable: it allows different probability to be attached to two off-path nodes that succeed the same information set and that are realised by the same action in that information set.

¹Perhaps players make mistakes, or perhaps they deliberately experiment...

3 Sequential Equilibrium

An assessment (σ, μ) is an SE iff:

[sequential rationality] (1) each σ_i is optimal given σ_{-i} and μ_i

[consistency] (2) there exists a sequence $\langle (\sigma^m, \mu^m) \rangle_{m=1}^{\infty}$ such that:

(2.1) (σ^m, μ^m) converges to (σ, μ) as $m \rightarrow \infty$

(2.2) σ^m is a completely mixed strategy profile for $\forall m$

(2.3) μ^m is computed via Bayes rule for $\forall m$

3.1 Existence of SE

If (1) there are finitely many terminal histories,

and (2) every terminal history is finite,

(‘finite time horizon’)

then: there exists at least one SE.

3.2 Finding SE

When finding SE, it’s usually easiest to find PBE first, and then consider which of them can be justified as SE.

4 Relationship Between PBE and SE

Whenever all players know their own payoff types (e.g. know whether the chance player gives them – at the start of the game – the ‘sane’ or the ‘crazy’ payoff type), PBE is equivalent to SE.

5 Intuitive Criterion

A player's beliefs-at-some-information-set within an SE can place positive probability on an opponent having played a *strictly dominated strategy*.²

The intuitive criterion is further refinement to SE designed to rule out such beliefs. In the context of games in which nature assigns players 'types': "upon observing a deviation, players form beliefs that put zero weight on types whose equilibrium payoff exceeds all possible payoffs from deviating."

The intuitive criterion seems to commit us to a particular story about why players deviate – one that excludes the possibility of 'trembling hands'.

6 Further Refinements

Many games have been found where refinements stronger than the Intuitive Criterion are needed to select the intuitively plausible equilibrium.

These games seem to push us further and further toward incorporating the logic of 'forward induction' into our solution concept

However, the 'holy grail' of equilibrium refinements has never been found. A solution concept that combines backward and forward induction on the part of players in fact looks to be inconsistent (there are 'impossibility theorems' that prove this, at least for certain ways of formalizing forward and backward induction).

²NB: Never playing a strictly dominated strategy is one the consequences of being capable of forward induction.

7 Forward Induction

Forward induction isolates play that is at each stage compatible with it being common knowledge that: past choices were rational responses to a certain belief system, that the chooser will continue to hold unless given reason to believe otherwise.

Forward induction relies heavily – unlike backward induction – on the idea that players form a certain belief system before making their first move.

7.1 Example

An example of forward induction is a revised version of the Battle of the Sexes, in which the man has the chance to publicly burn an amount of his utility before the BoS game begins, such that:

- (1) burning and playing cafe would be strictly dominated by not burning and playing pub, but
- (2) burning and achieving (pub, pub) would be preferable to not burning and achieving (cafe, cafe).

If this holds, then burning would be a signal to a woman capable of forward induction that the man expects her to play pub.

But note further that it is common knowledge that the man has the option to signal by playing burn, and that $(\text{burn, pub, pub}) \succ_{\text{man}} (\neg\text{burn, cafe, cafe})$. As such, it is common knowledge that if the man plays $\neg\text{burn}$, he must be expecting to get something better than (burn, pub, pub) , *viz.* $(\neg\text{burn, pub, pub})$. And because he will continue to expect that the woman will choose pub *until* he is given reason to believe otherwise, *this expectation will – if he has it – become self-fulfilling.*