# *Bayesian Games*

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## 1 Setup

### 1.1 Elements

| | |
|---|---|
| $\langle\ N$ | the set of players |
| $\{T_i\}_{i \in N}$ | for each player $i$, a set of *types* |
| $\{A_i\}_{i \in N}$ | for each player $i$, a set of actions *not* a function of $t_i$ |
| $\{p_i(\cdot\,\vert t_i)\}_{i \in N, t_i \in T_i}$ | for each player-type pair $\langle i, t_i \rangle$, a *belief* over $T_{-i}$ treat as primitive (rather than derived from a prior) |
| $\{u_i\}_{i \in N}\ \rangle$ | Bernoulli payoff function over $\{\langle a, t \rangle\}_{a \in A, t \in T}$ |

### 1.2 Knowledge

Each player knows:

1. their own type
2. the structure of the game[1]

### 1.3 Types

We use a player's type to specify all of their relevant characteristics:

$$\text{player } i\text{'s type includes} \begin{cases} \text{her preferences over } A \\ \\ \text{her information about} \\ \text{every other player's type.} \\ \text{player } j \neq i\text{'s type includes:} \begin{cases} \text{her preferences over } A \\ \\ \text{her information about} \\ \text{every other player's type.} \\ \text{player } k \neq j\text{'s type includes:} \begin{cases} \dots \end{cases} \end{cases} \end{cases}$$

[1]In game theory, we *always* assume that each player knows the structure of the game.

### 1.4 Nature

If we need to bring nature into the model, then we treat her as a player, with a type on whose value players' preferences will depend.

### 1.5 Choice

Each player chooses a **Bayesian strategy** $s_i(\cdot) : T_i \to A_i$.

Equivalently: each *type* of player chooses an action.

Players cannot choose to change their beliefs $p_i$: these are fixed by the game.

## 2 Notable Types of Bayesian Game

### 2.1 Global Games

Global games are Bayesian games in which players are uncertain of their *own payoffs* as well as their opponents. In global games, players' uncertainty takes the form: 'what *game am I playing* here?'.

The motivating idea behind the global games literatures is that sometimes *adding some uncertainty* to a game being played can significantly reduce the set of NE.

### 2.2 Continuous Strategy Spaces

In Bayesian games where each player has a continuous strategy space, we can often find an equilibrium by assuming that each player uses a **linear strategy**: $s_i(t_i) = \phi_i + \psi_i t_i$ .

# 3 Bayesian-Nash Equilibrium

NE predicts the steady state behaviour of a set of populations – one for each player – playing the game repeatedly against a randomly selected anonymous opponent each time.

Thus in an NE: the action chosen by each type $t_i$ of any player $i$ is optimal,

1. given the actions chosen by each type of every other player, and
2. given player $i$ type $t_i$'s *beliefs* $p_i(\cdot|t_i)$.

Since players cannot choose to change their beliefs $p_i$,[2] our definition of NE places no restrictions on them.

## 3.1 Formal Definition

The NE of a Bayesian game are the NE of the strategic game with:

| | | |
|---|---|---|
| *players* | the set of all pairs | $(i, t_i)$ |
| *actions* | for each $(i, t_i)$, | $A_{(i,t_i)} = A_i$ |
| *payoffs* | for each $(i, t_i)$, | $u_{(i,t_i)}(a)$ = the expected value – from the perspective of player $i$ type $t_i$ – of player $i$'s utility should she play $a_i$ (and the others play $a_{-i}$) |

## 3.2 Bayesian-Nash Equilibrium

Rephrasing the NE of a Bayesian game into *Bayesian strategies* gives us the Bayesian-Nash equilibrium.

## 3.3 Impossible Type-Profiles

Suppose that player $i$ type $t_i$ thinks that $t_{-i}$ happens with positive probability, but it is in fact impossible.

The NE actions of the players $j \neq i$ in that type-profile can be interpreted as player $i$'s correct beliefs about what actions the players $j \neq i$ *would* take in those circumstances.

[2]This aspect of the equilibrium concept may well strike you as surprising, or objectionable.

# 4 Harsanyi Purification

Harsanyi Purification is a way of approaching games in which:

1. Each player's payoff function is subject to small random disturbances with a given range.[3]
2. The probability laws governing these disturbances are known to all players.

## 4.1 Application to Mixed NE

Almost all mixed-strategy NE of a game are:

slightly imprecise descriptions of
a *strict* pure Bayesian-NE of
a slightly perturbed version of the game.

## 4.2 'Cutoff Strategies'

The Bayesian strategies yielded by Harsanyi purification will be **cutoff strategies:**

if $\varepsilon$ is the random variable perturbing player $i$'s payoffs, player $i$ will play $a_i^1$ iff $\varepsilon < \varepsilon^*$, and $a_i^2$ iff $\varepsilon \geq \varepsilon^*$.

## 4.3 The Procedure

Show that as $\varepsilon \to 0$, the Bayesian equilibrium $\to$ the mixed NE of the unperturbed game.

[3]E.g. due to stochastic fluctuations in *mood*.